Indian Statistical Institute, Bangalore
B. Math (Hons.) Second Year
First Semester - Optimization

Midterm Exam — Date: September 14, 2018
Maximum marks: 30 — Duration: 2.30 hours

Answer any five, each question carries 6 marks

1. Prove that for any $p \times q$ non-zero matrix A, there exists an invertible matrix G such that GA is upper echolen.

2. (a) Find the QR decomposition of $\begin{pmatrix} 1 & 0 & 2 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ *(Marks: 4)*.

   (b) Give a 3×3 example to show that QR decomposition is not unique.

3. (a) Prove that every $p \times q$-matrix has singular value decomposition.

   (b) Prove that the singular values of A and A^* coincide. *(Marks: 2)*.

4. (a) If $A = \text{diag}(r_1, \cdots, r_d)$, find $||A||$? (*Marks: 2*).

   (b) Let $A \in M_{p\times q}(\mathbb{C})$ and s_1 be the first singular value. Prove that $s_1 = ||A||$.

5. Let S be a subspace of $\mathbb{R}^n$ and $a, u \in \mathbb{R}^n$. Denote $W = a + S = \{a + x \mid x \in S\}$.

   (a) Prove that u can be written uniquely as $w + y$ for $w \in W$ and $y \in S^{\perp}$.

   (b) Prove that $\min_{y \in W} ||u - y||$ has a unique solution *(Marks: 3)*.

6. (a) Prove that $\text{spr}(A) \leq ||A||$ for any square matrix A.

   (b) Let A be a non-negative irreducible matrix. If λ is a eigenvalue of A with eigenvector $x \geq 0$, prove that λ is the spectral radius (*Marks: 4*).

7. Prove that $A = \begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 3 \\ 0 & 2 & 0 \end{pmatrix}$ is irreducible and find its Perron-pair. Is it primitive? Justify your answer.